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★**Les jeux de l'infini et du hasard. [Games of infinity and of chance]**
Vol. 1. Les probabilités dénombrables à la portée de tous. [Vol. 1. Countable probabilities accessible to all]
Vol. 2. Les probabilités indénombrables à la portée de tous. Annexes et appendices. [Vol. 2. Uncountable probabilities accessible to all. Additional texts and appendices]
Sciences: Concepts et Problèmes. [Sciences: Concepts and Problems]

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This book, in two volumes, is a history of probability written from a humanistic and intellectual perspective in discursive commentary style, with much attention to the details of the social and scientific backdrop of the protagonists Émile Borel (1871–1956) and Pierre-Simon de Laplace (1749–1827), as well as their enormous retinues. It also has the nature of an encyclopedia through its extremely useful Name Index (pp. 775–816), which also gives birth and death years. The pagination of the two volumes is consecutive: Volume 1, pp. 1–286; Volume 2, pp. 287–818.

The book arose out of the authors' promise to Marc Barbut (1928–2011) to celebrate the centenary of "Les probabilités dénombrables" of Borel [Rend. Circ. Mat. Palermo **27** (1909), 247–271; JFM 40.0283.01]. This is one of the foundational texts of probability in the 20th century, in its conception of an event describing a denumerably infinite number of events occurring. Other centenaries intervened, especially those of Laplace's *Théorie analytique des probabilités* (1812, 1814) [*Théorie analytique des probabilités. Vol. I*, reprint of the 1819 fourth edition (Introduction) and the 1820 third edition (Book I), Éd. Jacques Gabay, Paris, 1995; MR1400402; *Théorie analytique des probabilités. Vol. II*, reprint of the 1820 third edition (Book II) and of the 1816, 1818, 1820 and 1825 originals (Supplements), Éd. Jacques Gabay, Paris, 1995; MR1400403], so eventually the authors blended the Borel and Laplace material as the dominant content of the two volumes.

Laplace worked, rather, with a finite number of events, n, and used asymptotic methods of analysis as $n \to \infty$ such as Stirling's formula to develop versions of the celebrated theorem for limits of probabilities relating to the number of successes in binomial trials. The distinction between the approaches of Borel and Laplace is now reflected, respectively, in "convergence almost surely" and "convergence in distribution", since the latter is a concept in the convergence of monotonic functions, while the former requires measure-theoretic probability for its proper expression. The distinction between the two frameworks, denumerably infinite and finite, is the basis for the division of the book into two volumes.

Barbut, of the École des hautes études en sciences sociales, Paris, was the driving force behind the bilingual *Electronic Journal for the History of Probability and Statistics* (Laurent Mazliak, ed.), and especially (with himself as chief editor) of *Mathématiques et sciences humaines,* in the spirit of which the present two volumes are written. According to Barbut's wish, the book is intended "for everybody", especially for students of the probability calculus, to provide a diversity of historical trails to follow. Accordingly, the Bibliography (pp. 589–774) is enormous. And there is a didactic component, an introduction to the history of the probability calculus, as Annexes 1 and 2 of Volume 2. Annexe 2 is derived from [B. Bru, Math. Sci. Hum. Math. Soc. Sci. No. 175 (2006), 5–23; MR2275932], a delightful exposition not listed in the book's Bibliography. Annexe 1, on "Dice games", is a study in the prehistory of the problem of points, that is on the equitable division of stakes for an unfinished game of chance. This problem is associated in the history of probability with the names of Pascal and Fermat in 1654. The present authors take us through the calculations in the much older *De vetula*. This annex has already been translated into English by Glenn Shafer and published as [M.-F. Bru and B. Bru, Statist. Sci. **33** (2018), no. 2, 285–297; MR3797715]. Shafer's article [G. R. Shafer, Statist. Sci. **33** (2018), no. 2, 277–284; MR3797714], anticipating the imminent appearance of the long-awaited two-volume history of mathematical probability under review, is in itself a background to (and partial review of) the whole.

Borel [op. cit. (Chapter 1)] considered an infinite sequence of independent trials with probability p_k of success at the k-th trial, $k \geq 1$, and concluded that the probability of an infinite number of successes is 0 if $\sum p_k < \infty$, and is 1 if $\sum p_k = \infty$. This has come to be known as "Borel's zero-one law". He himself called it "the fundamental theorem of the theory of denumerable probabilities". His proof is deficient in a number of respects, and his attempt to use the result to deduce a Strong Law of Large Numbers was inappropriate since the events considered there are not independent. The result, which is called the Borel-Cantelli Lemma in now-classical axiomatic probability theory, reads: For any sequence of events $\{A_n\}$ in a given probability space $(\Omega, \mathcal{B}, P)$, where Ω is the sample space, $\mathcal{B}$ is a σ-algebra of sets of sample points, called "events", and P is a σ-additive probability measure on $\mathcal{B}$, then

$$\sum_n P(A_n) < \infty \Rightarrow P(A_n \text{ i.o.}) = 0$$

(this is the "convergence" part). If the events $\{A_n\}$ are independent, then

$$\sum_n P(A_n) = \infty \Rightarrow P(A_n \text{ i.o.}) = 1$$

(this is the "divergence" part). Consequently, this result is ascribed to Borel for independent events, and to Cantelli, in 1917, in the "convergence" version for not necessarily independent events. Here $\{A_n \text{ i.o.}\}$ is the event that an infinite number of the events $\{A_n\}$ occur. There is a very large literature on these matters, which is fully cited within the immense Bibliography of the book under review, and discussed in meticulous detail in Volume 1, Section 7 (pp. 71–94), which covers Borel's work on "probabilités denombrables" of 1896–1909; and also in the extensive notes (125), pp. 193–194, and (159), p. 239. For the English-language reader, a lead-in to the literature and issues is in [E. Seneta, Historia Math. **19** (1992), no. 1, 24–39; MR1150880].

Borel's preoccupation in 1909–1949 with "martingales dénombrables", presented in Volume 1, Section 8, relates to the St. Petersburg Paradox, which originated in 1713. Here "the game stops" can occur at any positive integer time point. The material was presented more formidably in [B. Bru, M.-F. Bru and K. L. Chung, J. Électron. Hist. Probab. Stat. **5** (2009), no. 1, 58 pp.; MR2520662], but the introductory first part of Section 8 is on Laplace's dismissal, in his *Essai philosophique des probabilités* of 1816 [see op. cit.; MR1400402], of two martingales focussing on strategies for perpetuating the family surname, by males procreating until a son, or two successive sons, are born. This section testifies to the authors' detailed knowledge of, and devotion to, Laplace, to whom the 8 Appendices, pp. 377–588 in Volume 2, are devoted.

The authors write in their introduction to Volume 2 that these appendices "turn about the personnage of Laplace and his *Théorie analytique* which remains one of the summits of mathematical literature of all times". Appendix 1 (pp. 377–394) accordingly describes "the method of Laplace for the approximation of formulae which are functions of very large numbers". The centerpiece result of Laplace to which the methodology finds application is the "inverse" of the Moivre-Laplace Theorem, the "direct" theorem which gives the standard normal distribution (the Gaussian curve) for the limiting distribution as $n \to \infty$ of the standardized proportion of successes, P, observed in n binomial trials with fixed probability of success θ. The inverse theorem presupposes a prior uniform distribution on $(0, 1)$ for θ, and finds that the standard normal probabilities can be used to describe approximately, for large n, the posterior distribution of θ, given the number of successes. Thus, in regard to modern mathematical statistics, Laplace was a Bayesian, and his technical development in this direction is competently and effectively described in the "encyclopedias of mathematical statistics" of S. M. Stigler [*The history of statistics*, Belknap Press/Harvard Univ. Press, Cambridge, MA, 1986; MR0852410] and A. H. Hald [*A history of mathematical statistics from 1750 to 1930*, Wiley Ser. Probab. Stat. Texts Ref. Sect., Wiley, New York, 1998; MR1619032]. Indeed, Bayesian inference, from effect to cause, was the only form of inference about parameters, and as Augustus De Morgan (1806–1871) observed at the end of his 1837 two-part review of the third (1820) edition of the *Théorie analytique*: "the greater part of the treatise is full of ... questions ... bearing in the most direct manner on the way to draw correct inferences from physical and statistical facts". In fact, two of Laplace's major and interrelated statistical themes are, in modern terms, the Central Limit Theorem expressed in the form of the Gaussian curve, and his justification of the method of least squares, with the Gaussian distribution as a Law of Error. It is from De Morgan's review that the description of the *Théorie analytique* as "the Mont Blanc of mathematical analysis" comes, whose quotation continues, however, with "it gave neither finish nor beauty to the results".

In 1837 De Morgan also published a lengthy article entitled *Theory of probabilities* in the *Encyclopedia Metropolitana*, which served not only as a digest of the *Théorie analytique*, but also a simplification and clarification of the proofs, and gave rise to a British "Laplacian" direction [A. C. Rice and E. Seneta, J. Roy. Statist. Soc. Ser. A **168** (2005), no. 3, 615–627; MR2146412]. Similarly, Viktor Yakovlevich Buniakovsky (1804–1889), who had studied at the Sorbonne and College de France in Paris from 1824 to 1825, and had contact with the work of Laplace, Poisson, Fourier, Legendre, and Ampère, but principally Cauchy (a contact commemorated in the Cauchy-Schwarz-Buniakovsky Inequality), presented to the St. Petersburg Academy in 1841 an announcement entitled *Sur la publication, en russe, d'une Théorie analytique des probabilités*. This shows clearly his intention to parallel in nature Laplace's treatise, with the aim of filling a gap due to the absence of probability theory in the Russian mathematical literature, while making "more accessible the delicate theories of which the calculus of chances offers many examples". The resulting book was *Osnovaniia matematecheskoi teorii veroiatnostei [Foundations of the mathematical theory of probabilities]* (1846). The book separates out *a priori* calculation of probabilities from *a posteriori*, and presents Buniakovsky as a disciple and elucidator of Laplace in probability theory, similar to De Morgan in the British setting and Irenée-Jules Bienaymé [C. C. Heyde and E. Seneta, *I. J. Bienaymé. Statistical theory anticipated*, Springer, New York, 1977; MR0462888] and possibly Hermann Laurent, in the French, while creating Russian-language probabilistic terminology. A Bayesian tradition of statistical inference lived on in the Russian Empire up to the Bolshevik revolution, including some writings of A. A. Markov.

Appendix 6 is about Hermann Laurent's *Théorie analytique* (1873).

Appendix 2, on the "statistical geometry" of Laplace (1776–1812), seeks to connect this topic with the later work of Borel on the calculation of volumes in an arbitrary number of dimensions. It describes how Laplace was motivated to return to probability theory "proper" in 1810 by geometric considerations in cosmology, and indeed, consequently to formulate "Laplace's Theorem". Borel was aware of the similar German-language work of Sommerfeld and Pólya, but, like these authors, was unaware that some of their material was present already in the *Théorie analytique*. Their geometric subject matter was based on the volume defined by

$$|x_i| \le a_i,\ i = 1, \dots, n; \quad \left| \sum_{i=1}^{n} x_i \right| \le a_{n+1},$$

for which Pólya found an elegant integral form which in the case $a_i = \frac{1}{2}$, $a_{n+1} = \eta\sqrt{n}$ is in fact $P(|\sum_{i=1}^{n} X_i| \le \eta\sqrt{n})$, for independently and uniformly distributed random variables X_i, $i = 1, \dots, n$ on $(-\frac{1}{2}, \frac{1}{2})$. Then allowing $n \to \infty$ in this expression gives a limiting normal distribution of variance $\frac{1}{12}$.

Appendices 7 and 8 are consequently about the work of Sommerfeld (1904) and Pólya (1912), which was in the Gaussian, rather than the Laplacian, direction.

Appendix 3 is on the inversion formulae of Lagrange.

Appendix 4, entitled “Propaganda laplacienne”, addresses the paradox of the *Théorie analytique* and *Essai philosophique* being both the epitomy of scientific and philosophical creativity, and being mostly unreadable; and unread after the first half of the 19th century until the Stigler and Hald treatises. It begins with Laplace’s own beliefs in the works, and the propagation of these convictions.

Appendix 5, entitled “Souvenirs laplaciennes”, looks at aspects of Laplace’s interests which don’t relate directly to the probability calculus. Laplace lived through a stormy period of French history, when, for example, religion was tightly controlled. A section dealing with his family life is fascinating.

This book is an essential for any student of the history and sociology of probability, and of mathematical statistics in general. That subject matter is deeply imbedded in the French tradition. These volumes, with the aid of the excellent name index, are full of unexpected gems of information and insight within a rich tapestry of history. The work is clearly a kind of swansong, an illuminating culmination of many years of deep study and devotion. *E. Seneta*